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Two exact sequences for lattice cohomology

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Dedicated to Henri Moscovici on his 60th birthday.

Abstract. This article is a continuation of [**N08**], where the lattice cohomology of connected negative definite plumbing graphs was introduced. Here we consider the more general situation of non–degenerate plumbing graphs, and we establish two exact sequences for their lattice cohomology. The first is the analogue of the surgery exact triangle proved by Ozsváth and Szabó for the Heegaard–Floer invariant HF^+; for the lattice cohomology over $\mathbb{Z}_2$–coefficients it was proved by J. Greene in [**Gr08**]. Here we prove it over $\mathbb{Z}$, and we supplement it by some additional properties valid for negative definite graphs. The second exact sequence is an adapted version which does not mix the classes of the characteristic elements ($spin^c$–structures); it was partially motivated by the surgery formula for the Seiberg–Witten invariant obtained in [**BN10**]. For this, we define the *relative lattice cohomology* and we determine its Euler characteristic in terms of Seiberg–Witten invariants.

1. Introduction

The lattice cohomology $\{\mathbb{H}^q(\Gamma)\}_{q\geq 0}$ was introduced in [**N08**]. In its original version, it was associated with any connected negative definite plumbing graph Γ, or, equivalently, with any oriented 3–manifold, which might appear as the link of a local complex normal surface singularity. Lattice cohomology (together with the graded roots) plays a crucial role in the comparison of the analytic and topological invariants of surface singularities, cf. [**N05, N07, N08**], see also [**BN10, NN02, N10**] for relations with the Seiberg–Witten invariants of the link. Additionally, the lattice cohomology (conjecturally) offers a combinatorial description for the Heegaard–Floer homology of Ozsváth and Szabó (for this theory see [**OSz03, OSz04, OSz04b**] and the long list of articles of Ozsváth and Szabó). Indeed, in [**N08**] the author conjectured that $\oplus_{q\ even/odd}\mathbb{H}^q(\Gamma)$ is isomorphic as a graded $\mathbb{Z}[U]$–module with $HF^+_{even/odd}(-M(\Gamma))$, where $M(\Gamma)$ is the plumbed 3–manifold associated with Γ. (Recall that at this moment there is no combinatorial definition/characterization of HF^+.)

For rational and 'almost rational' graphs this correspondence was established in [**OSz03, N05**] (see also [**NR10**] for a different situation and [**N08**] for related results). A possible machinery which might help to prove the general conjecture is based on the surgery exact sequences. They are established for the Heegaard–Floer theory in the work of Ozsváth and Szabó. Our goal is to prove the analogous exact sequences for the lattice cohomology. In fact, independently of the above conjecture and correspondence, the proof of such exact sequences is of major importance, and they are fundamental in the computation and in finding the main properties of the lattice cohomologies.

The formal, combinatorial definition of the lattice cohomology permits to extend its definition to arbitrary graphs (plumbed 3–manifolds), the connectedness and negative definiteness assumptions can be dropped. Nevertheless, in the proof of the exact sequences, we will deal only with non–degenerate graphs (they are those graphs whose associated intersection form has non–zero determinant).

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More precisely, for any graph Γ and fixed vertex j_0, we consider the graphs $\Gamma\setminus j_0$ and $\Gamma^+_{j_0}$, where the first one is obtained from Γ by deleting the vertex j_0 and adjacent edges, while the second one is obtained from Γ by replacing the decoration e_{j_0} of the vertex j_0 by $e_{j_0}+1$. We will assume that all these graphs are non–degenerate. Then Theorem 5.5.3 establishes the following long exact sequence.

Theorem A. *Assume that the graphs $\Gamma^+_{j_0}$, Γ and $\Gamma\setminus j_0$ are non–degenerate. Then*

$$\cdots\longrightarrow \mathbb{H}^q(\Gamma^+_{j_0})\xrightarrow{\mathbb{A}^q}\mathbb{H}^q(\Gamma)\xrightarrow{\overline{\mathbb{B}}^q}\mathbb{H}^q(\Gamma\setminus j_0)\xrightarrow{\mathbb{C}^q}\mathbb{H}^{q+1}(\Gamma^+_{j_0})\longrightarrow\cdots$$

is an exact sequence of $\mathbb{Z}[U]$–modules.

The first 3 terms of the exact sequence (i.e. the $\mathbb{H}^0$–part) were already used in [**OSz03**] (see also [**N05**]), and the existence of the long exact sequence was already proved over $\mathbb{Z}_2$–coefficients in [**Gr08**]. Here we establish its validity over $\mathbb{Z}$. In the proof we not only find the correct sign–modifications, but we also replace some key arguments. (Nevertheless, the proof follows the main steps of [**Gr08**].)

For negative definite graphs (i.e. when $\Gamma^+_{j_0}$, hence Γ and $\Gamma\setminus j_0$ too are negative definite), the above exact sequence has some important additional properties. By general theory (cf. [**N08**]), if Γ is negative definite then $\mathbb{H}^0(\Gamma)$ contains a canonical submodule $\mathbb{T}$ and one has a direct sum decomposition $\mathbb{H}^0=\mathbb{T}\oplus\mathbb{H}^0_{red}$. $\mathbb{T}$ is the analogue of the image of HF^∞ in HF^+ in the Heegaard–Floer theory. On the other hand, in Heegaard–Floer theory by a result of Ozsváth and Szabó, the operator $\mathbb{C}^0$ restricted on $\mathbb{T}(\Gamma\setminus j_0)$ is zero (this follows from the fact that the corresponding cobordism connecting $\Gamma\setminus j_0$ and $\Gamma^+_{j_0}$ is coming from a non–negative definite surgery). The analogue of this result is Theorem 6.1.2:

Theorem B. *Assume that $\Gamma^+_{j_0}$ is negative definite. Consider the exact sequence*

$$0\longrightarrow \mathbb{H}^0(\Gamma^+_{j_0})\xrightarrow{\mathbb{A}^0}\mathbb{H}^0(\Gamma)\xrightarrow{\overline{\mathbb{B}}^0}\mathbb{H}^0(\Gamma\setminus j_0)\xrightarrow{\mathbb{C}^0}\mathbb{H}^1(\Gamma^+_{j_0})\longrightarrow\cdots$$

and the canonical submodule $\mathbb{T}(\Gamma\setminus j_0)$ of $\mathbb{H}^0(\Gamma\setminus j_0)$. Then the restriction $\mathbb{C}^0|\mathbb{T}(\Gamma\setminus j_0)$ is zero.

Using the above results one proves for a graph Γ *with at most n bad vertices* (for the definition, see 6.2) the vanishing $\mathbb{H}^q_{red}(\Gamma)=0$ for any $q\geq n$ (where $\mathbb{H}^q_{red}=\mathbb{H}^q$ for $q>0$).

$\mathbb{H}^*(\Gamma)$ has a natural direct sum decomposition indexed by the set of the characteristic element classes (in the case of the Heegaard–Floer, or Seiberg–Witten theory, they correspond to the *spin^c*–structures of $M(\Gamma)$). Namely, $\mathbb{H}^*(\Gamma)=\oplus_{[k]}\mathbb{H}^*(\Gamma,[k])$. In the exact sequence of Theorem A the operators mix these classes. Theorem 7.2.2 provides an exact sequence which connects the lattice cohomologies of Γ and $\Gamma\setminus j_0$ with fixed (un–mixed) characteristic element classes. More precisely, for any characteristic element k of Γ, we define a $\mathbb{Z}[U]$–module $\{\mathbb{H}^q_{rel}(k)\}_{q\geq 0}$, the *relative lattice cohomology* associated with (Γ,j_0,k). It has finite $\mathbb{Z}$–rank and it fits in the following exact sequence:

Theorem C. *Assume that Γ and $\Gamma\setminus j_0$ are non–degenerate. One has a long exact sequence of $\mathbb{Z}[U]$–modules:*

$$\cdots\longrightarrow \mathbb{H}^q_{rel}(k)\xrightarrow{\mathbb{A}^q_{rel}}\mathbb{H}^q(\Gamma,[k])\xrightarrow{\mathbb{B}^q_{rel}}\mathbb{H}^q(\Gamma\setminus j_0,[R(k)])\xrightarrow{\mathbb{C}^q_{rel}}\mathbb{H}^{q+1}_{rel}(k)\longrightarrow\cdots$$

where $R(k)$ is the restriction of k (and the operators also depend on the choice of the representative k).

The existence of such a long exact sequence is predicted and motivated by the surgery formula for the Seiberg–Witten invariant established in [**BN10**]: the results of section 7 resonate perfectly with the corresponding statements of Seiberg–Witten theory. This allows us to compute the Euler characteristic of the relative lattice cohomology in terms of the Seiberg–Witten invariants associated with $M(\Gamma)$ and $M(\Gamma\setminus j_0)$.

2. Notations and preliminaries

2.1. Notations. First we will introduce the needed notations regarding plumbing graphs and we recall the definition of the *lattice cohomology* from [**N08**]. Since in [loc.cit.] the graphs were *connected* and *negative definite* (as the 'normal' plumbing representations of isolated complex surface singularity links), at the beginning *we will start with these assumptions*.

Let Γ be such a plumbing graph with vertices $\mathcal{J}$ and edges $\mathcal{E}$; we set $|\mathcal{J}|=s$. and we fix an order on $\mathcal{J}$. Γ can also be codified in the lattice L, the free $\mathbb{Z}$–module generated by $\{E_j\}_{j\in\mathcal{J}}$ and the 'intersection form' $\{(E_i,E_j)\}_{i,j}$, where (E_i,E_j) for $i\neq j$ is 1 or 0 corresponding to the fact that (i,j) is an edge or not; and (E_i,E_i) is the decoration of the vertex i, usually denoted by e_i. (The graph is negative definite if this form is so.) The graph Γ may have cycles, but we will assume that all the genus decorations are zero (i.e. we plumb S^1–bundles over S^2). The associated plumbed 3–manifold $M(\Gamma)$ is not necessarily a rational homology

sphere, this happens exactly when the graph is a tree. Let L' be the dual lattice $\{l' \in L\otimes\mathbb{Q} : (l', L)\subset \mathbb{Z}\}$; it is generated by $\{E^*_j\}_j$, where $(E^*_j, E_i) = -\delta_{ij}$ (the *negative* of the Kronecker–delta). Moreover,

$$Char := \{k\in L' : \chi_k(l) := -\frac{1}{2}(k+l,l)\in\mathbb{Z} \ \text{ for all } l\in L\}$$

denotes the set of characteristic elements of L (or Γ).

As usual (following Ozsváth and Szabó), $\mathcal{T}^+_0$ denotes the $\mathbb{Z}[U]$–module $\mathbb{Z}[U,U^{-1}]/U\mathbb{Z}[U]$ with grading $\deg(U^{-d}) = 2d$ ($d\geq 0$). More generally, for any $r\in\mathbb{Q}$ one defines $\mathcal{T}^+_r$, the same module as $\mathcal{T}^+_0$, but graded (by $\mathbb{Q}$) in such a way that the $d+r$–homogeneous elements of $\mathcal{T}^+_r$ are isomorphic with the d–homogeneous elements of $\mathcal{T}^+_0$. (E.g., for $m\in\mathbb{Z}$, $\mathcal{T}^+_{2m} = \mathbb{Z}[U,U^{-1}]/U^{-m+1}\mathbb{Z}[U]$.)

2.2. The lattice cohomology associated with $k\in Char$ [N08]. $L\otimes\mathbb{R} = \mathbb{Z}^s\otimes_{\mathbb{Z}}\mathbb{R}$ has a natural cellular decomposition into cubes. The set of zero–dimensional cubes is provided by the lattice points L. Any $l\in L$ and subset $I\subset\mathcal{J}$ of cardinality q defines a q–dimensional cube, which has its vertices in the lattice points $(l+\sum_{j\in I'} E_j)_{I'}$, where I' runs over all subsets of I. On each such cube we fix an orientation. This can be determined, e.g., by the order $(E_{j_1},\ldots,E_{j_q})$, where $j_1<\cdots<j_q$, of the involved base elements $\{E_j\}_{j\in I}$. The set of oriented q–dimensional cubes defined in this way is denoted by $\mathcal{Q}_q$ ($0\leq q\leq s$).

Let $\mathcal{C}_q$ be the free $\mathbb{Z}$–module generated by oriented cubes $\square_q\in\mathcal{Q}_q$. Clearly, for each $\square_q\in\mathcal{Q}_q$, the oriented boundary $\partial\square_q$ has the form $\sum_k\varepsilon_k\,\square^k_{q-1}$ for some $\varepsilon_k\in\{-1,+1\}$. Here, in this sum, we write only those $(q-1)$–cubes which appear with non–zero coefficient. One sees that $\partial\circ\partial = 0$, but, obviously, the homology of the chain complex $(\mathcal{C}_*,\partial)$ is trivial: it is just the homology of $\mathbb{R}^s$. In order to get a more interesting (co)homology, one needs to consider a *weight functions* $w:\mathcal{Q}_q\to\mathbb{Z}$ ($0\leq q\leq s$). In the present case this will be defined, for each $k\in Char$ fixed, by

$$w(\square_q) := \max\{\chi_k(l) \ : \ l \ \text{ is a vertex of } \square_q\}.$$

Once the weight function is defined, one considers $\mathcal{F}^q$, the set of morphisms $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{C}_q,\mathcal{T}^+_0)$ with finite support on $\mathcal{Q}_q$. Notice that $\mathcal{F}^q$ is, in fact, a $\mathbb{Z}[U]$–module by $(p*\phi)(\square_q) := p(\phi(\square_q))$ ($\phi\in\mathcal{F}^q$, $p\in\mathbb{Z}[U]$). Moreover, $\mathcal{F}^q$ has a $2\mathbb{Z}$–grading: $\phi\in\mathcal{F}^q$ is homogeneous of degree $2d\in\mathbb{Z}$ if for each $\square_q\in\mathcal{Q}_q$ with $\phi(\square_q)\neq 0$, $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}^+_0$ of degree $2d-2\cdot w(\square_q)$.

Next, one defines $\delta:\mathcal{F}^q\to\mathcal{F}^{q+1}$. For this, fix $\phi\in\mathcal{F}^q$ and we show how $\delta(\phi)$ acts on a cube $\square_{q+1}\in\mathcal{Q}_{q+1}$. First write $\partial\square_{q+1} = \sum_k\varepsilon_k\square^k_q$, then set

$$(\delta(\phi))(\square_{q+1}) := \sum_k \varepsilon_k\, U^{w(\square_{q+1})-w(\square^k_q)}\,\phi(\square^k_q). \tag{2.2.1}$$

One verifies that $\delta\circ\delta = 0$, i.e. $(\mathcal{F}^*,\delta)$ is a cochain complex (with δ homogeneous of degree zero); its homology is denoted by $\{\mathbb{H}^q(\Gamma,k)\}_{q\geq 0}$.

$(\mathcal{F}^*,\delta_w)$ has a natural augmentation too. Indeed, set $m_k := \min_{l\in L}\{\chi_k(l)\}$. Then one defines the $\mathbb{Z}[U]$–linear map $\epsilon:\mathcal{T}^+_{2m_k}\longrightarrow\mathcal{F}^0$ such that $\epsilon(U^{-m_k-s})(l)$ is the class of $U^{-m_k+\chi_k(l)-s}$ in $\mathcal{T}^+_0$ for any integer $s\geq 0$. Then ϵ is injective and homogeneous of degree zero, $\delta\circ\epsilon = 0$. The homology of the augmented cochain complex

$$0\longrightarrow\mathcal{T}^+_{2m_k}\xrightarrow{\ \epsilon\ }\mathcal{F}^0\xrightarrow{\ \delta\ }\mathcal{F}^1\xrightarrow{\ \delta\ }\ldots$$

is called the *reduced lattice cohomology* $\mathbb{H}^*_{red}(\Gamma,k)$. For any $q\geq 0$, both $\mathbb{H}^q$ and $\mathbb{H}^q_{red}$ admit an induced graded $\mathbb{Z}[U]$–module structure and $\mathbb{H}^q = \mathbb{H}^q_{red}$ for $q>0$.

Note that ϵ provides a canonical embedding of $\mathcal{T}^+_{2m_k}$ into $\mathbb{H}^0$. Moreover, one has a graded $\mathbb{Z}[U]$–module isomorphism $\mathbb{H}^0 = \mathcal{T}^+_{2m_k}\oplus\mathbb{H}^0_{red}$, and $\mathbb{H}^*_{red}$ has finite $\mathbb{Z}$–rank.

Remark 2.2.2. For the definition of the lattice cohomology for more general weight functions and graphs with non–zero genera, see [**N08**].

2.3. Reinterpretation of the lattice cohomology. If $k' = k+2l$ for some $l\in L$ then $\mathbb{H}^*(\Gamma,k)$ and $\mathbb{H}^*(\Gamma,k')$ are isomorphic up to a degree–shift, cf. [**N08**, (3.3)]. In fact, all the modules $\{\mathbb{H}^*(\Gamma,k)\}_k$ can be packed into only one object, more in the spirit of [**OSz03**]. This was used in [**Gr08**] too.

In this way, for any fixed $k\in Char$, L is identified with the sublattice $k+2L\in Char$, and with the notation $l' := k+2l$ one has $\chi_k(l) = -\frac{1}{8}(l',l')+\frac{1}{8}(k,k)$. In particular, up to a shift in degree, for each fixed k, $l\mapsto\chi_k(l)$ and $l'\mapsto -\frac{1}{8}(l',l')$ define the same weight–function (on the cubes, see below), hence the same cohomology. In fact, we will modify even this weight function by $s/8$ (in this way the blowing up will induce a degree preserving isomorphism, cf. §3), and set:

$$w: Char\to\mathbb{Q}, \qquad w(k) := -\frac{k^2+s}{8} \qquad (s = |\mathcal{J}|). \tag{2.3.1}$$

The q–cubes $\square_q \in \mathcal{Q}_q$ are associated with pairs $(k, I) \in Char \times \mathcal{P}(\mathcal{J})$, $|I| = q$, (here $\mathcal{P}(\mathcal{J})$ denotes the power set of $\mathcal{J}$), and have the form $\{k + 2\sum_{j\in I'} E_j)_{I'}$, where I' runs over all subsets of I. The weights are defined by

$$w(\square_q) = w((k, I)) = \max_{I'\subset I} \{\, w(k + 2\sum_{j\in I'} E_j)\,\}. \tag{2.3.2}$$

Moreover, $\mathcal{F}^q$ are elements of $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{Q}_q, \mathcal{T}_0^+)$ with finite support. Similarly as above, $\mathcal{F}^q$ is a $\mathbb{Z}[U]$–module with a $\mathbb{Q}$–grading: $\phi \in \mathcal{F}^q$ is homogeneous of degree r if for each $\square_q \in \mathcal{Q}_q$ with $\phi(\square_q) \neq 0$, $\phi(\square_q)$ is a homogeneous element of $\mathcal{T}_0^+$ of degree $r - 2\cdot w(\square_q)$.

It is convenient to consider the module of (infinitely supported) homological cycles too: let $\mathcal{F}_q$ be the direct product of $\mathbb{Z}_{\geq 0}\times \mathcal{Q}_q$ copies of $\mathbb{Z}$ (considered already in [**OSz03**] for $q = 0$). We write the pair $(m, \square)$ as $U^m\square$. $\mathcal{F}_q$ becomes a $\mathbb{Z}[U]$–module by $U(U^m\square) = U^{m+1}\square$. Clearly $\mathcal{F}^q = \mathrm{Hom}_{\mathbb{Z}[U]}(\mathcal{F}_q, \mathcal{T}_0^+)$, i.e. $\phi(U\square) = U\phi(\square)$ for any ϕ.

$\delta : \mathcal{F}^q \to \mathcal{F}^{q+1}$ is defined as in (2.2.1) using the new weight–function, or by $\delta(\phi)(\square) = \phi(\partial(\square))$, where for $\square = (k, I) = (k, \{j_1, \ldots, j_q\})$ one has:

$$\partial(k, I) = \sum_{l=1}^{q} (-1)^l \big(U^{w(k,I)-w(k,I\setminus j_l)}(k, I \setminus j_l) - U^{w(k,I)-w(k+2E_{j_l},I\setminus j_l)}(k + 2E_{j_l}, I \setminus j_l) \big). \tag{2.3.3}$$

The cohomology of $(\mathcal{F}^*, \delta)$ is denoted by $\mathbb{H}^*(\Gamma)$. Since the vertices of a cube belong to the same class $Char/2L$ (where a class has the form $[k] = \{k + 2l\}_{l\in L} \subset Char$), $\mathbb{H}^*(\Gamma)$ has a natural direct sum decomposition

$$\mathbb{H}^*(\Gamma) = \oplus_{[k]\in Char/2L}\ \mathbb{H}^*(\Gamma, [k]).$$

In fact, if $[k_1] = [k_2]$ then $w(k_1) - w(k_2) \in \mathbb{Z}$.

Since Γ is negative definite, for each class $[k] \in Char/2L$ one has a well–defined rational number

$$d[k] := -\max_{k\in[k]} \frac{k^2 + s}{4} = 2\cdot \min_{k\in[k]} w(k).$$

Then (cf. 2.2) one has a direct sum decomposition:

$$\mathbb{H}^0(\Gamma, [k]) = \mathcal{T}^+_{d[k]} \oplus \mathbb{H}^0_{red}(\Gamma, [k]). \tag{2.3.4}$$

Sometimes we write $\mathbb{T} := \oplus_{[k]}\mathcal{T}^+_{d[k]}$ for the canonical submodule of $\mathbb{H}^0$, which satisfies $\mathbb{H}^0 = \mathbb{T} \oplus \mathbb{H}^0_{red}$.

Following [**N05, N08, N10**] we define the Euler characteristic of $\mathbb{H}^*(\Gamma, [k])$ by

$$eu(\mathbb{H}^*(\Gamma, [k])) := -d[k]/2 + \sum_{q\geq 0} (-1)^q\, \mathrm{rank}_{\mathbb{Z}}\ \mathbb{H}^q_{red}(\Gamma, [k]). \tag{2.3.5}$$

Remark 2.3.6. For any $\phi \in \mathcal{F}^q$ and $\ell \geq 0$ set $U^{-\ell} * \phi \in \mathcal{F}^q$ defined as follows: if $\phi(\square) = \sum_{m\geq 0} a_{m,\square}U^{-m}$, then $(U^{-\ell} * \phi)(\square) = \sum_{m\geq 0} a_{m,\square}U^{-m-\ell}$. Notice that $U(U^{-1} * \phi) = \phi$ (but, in general, $U^{-1} * (U\phi) \neq \phi$).

For any $\square \in \mathcal{Q}_q$ let $\square^\vee$ be that element of $\mathcal{F}^q$ which sends $\square$ in $1 \in \mathcal{T}_0^+$ and any other element into zero. Then any element of $\mathcal{F}^q$ is a finite $\mathbb{Z}$–linear combination of elements of type $U^{-\ell} * \square^\vee$.

2.4. Generalizations. Graphs which are not negative definite. Since the definition of the lattice cohomology is purely algebraic/combinatorial, its definition can be considered for any graph, or even for any lattice L with fixed base elements $\{E_j\}_j$. Nevertheless, in this note we assume that the graphs are non–degenerate. Of course, doing this generalization, some of the properties of the lattice cohomology associated with connected negative definite graphs will not survive. Here we wish to point out some of the differences.

First, we notice that dropping the connectedness of the graph basically has no effect (this fact already was used in the main inductive proofs of [**OSz03**] and [**N05**, (8.3)]). Dropping the negative definiteness is more serious. In order to explain this, we will introduce the following spaces: for any fixed class $[k]$ and any real number r we define S_r as the union of all cubes $\square \in \mathcal{Q}_q$ with all vertices in $[k]$ and $w(\square) \leq r$.

If Γ is negative definite then S_r is compact for any r. In particular, the weight function takes its minimum on it, and for each class $[k] = Char/2L$ a decomposition $\mathbb{H}^0(\Gamma, [k]) = \mathcal{T}^+_{d([k])} \oplus \mathbb{H}^0_{red}(\Gamma, [k])$ can be defined. This property will not survive for general graphs. In fact, if the components of S_r are not compact, then we even might have the vanishing $\mathbb{H}^0(\Gamma) = 0$ (and this can happen simultaneously with the non–vanishing of $\mathbb{H}^1$), see e.g. (2.4.1). Also, in [**N08**, Theorem 3.1.12] the lattice cohomology is recovered from the simplicial cohomology of the spaces S_r. In the general case, the adapted version of that theorem (with similar proof), valid for any lattice, states that the same formula is valid once if we replace the cohomology groups $H^q(S_r, \mathbb{Z})$ by the cohomology groups with compact support $H^q_c(S_r, \mathbb{Z})$. Namely:

$$\mathbb{H}^*(\Gamma, [k]) \simeq \bigoplus_{r\in w(k)+\mathbb{Z}} H^*_c(S_r, \mathbb{Z}).$$

We wish to emphasize, that replacing the cohomology with cohomology with compact support is a conceptual modification: the two groups have different functorial properties.

In fact, exactly this second point of view is determinative in the definition of the lattice cohomology (and in the definition of the maps in the surgery formula treated next): basically we mimic the infinitely supported homology and the (dual) cohomology with compact support, and the corresponding functors associated with them.

Since the surgery exact sequence considered in the next section is valid for any non–degenerate lattice, it is very convenient to extend the theory to arbitrary graphs (or, at least for non–degenerate ones) in order to have a larger flexibility for computations. Nevertheless, we have to face the following crucial problem. An oriented plumbed 3–manifold $M(\Gamma)$ has many different plumbing representations Γ. They are connected by the moves of the (oriented) plumbing calculus. For negative definite graphs the only moves are the blowups (and their inverses) by (rational) (-1)-vertices. In [**N08**, (3.4)] it is proved that the lattice cohomology is stable with respect to these moves, see also §3 here. On the other hand, if we enlarge our plumbing graphs, this stability condition will not survive: the same 3–manifold can be represented by many different lattices with rather different lattice cohomologies.

More precisely, for negative definite graphs one conjectures (cf. [**N08**]) that from the lattice cohomology one can recover (in a combinatorial way) the Heegaard–Floer homology of Ozsváth–Szabó. In particular, the lattice cohomology carries a geometric meaning depending only on $M(\Gamma)$. This geometric meaning is lost in the context of general graphs (or, at least, it is not so direct).

Example 2.4.1. S^3 can be represented by a graph with one vertex, which has decoration -1. Computing the lattice cohomology of this graph we get $\mathbb{H}^q = 0$ for $q \neq 0$ and $\mathbb{H}^0 = \mathbb{H}^0_{red} = \mathcal{T}^+_0$. (This is the Heegaard–Floer homology $HF^+(S^3)$.) On the other hand, it is instructive to compute the lattice cohomology of another graph, which has one vertex, but now with decoration $+1$. Its lattice cohomology is $\mathbb{H}^q = 0$ for $q \neq 1$ and $\mathbb{H}^1 = \mathcal{T}^+_{-1/2}$. This graph also represent S^3, and the two graphs can be connected by a sequence of non–empty graphs and ±1–blow ups and downs. In particular, we conclude that the lattice cohomology is not stable with respect to blowing up/down $(+1)$–vertices.

Remark 2.4.2. Replacing negative definite graphs with arbitrary non–degenerate ones we can also adopt the definition of the weight function (2.3.1) by modifying into

$$w_\dagger : Char \to \mathbb{Q}, \qquad w_\dagger(k) := \frac{-k^2 + 3\sigma + 2s}{8} \qquad (\sigma = \text{signature of } (\,,\,), \quad s = |\mathcal{J}|), \tag{2.4.3}$$

a more common expression associated with (4–manifold intersection) forms which are not negative definite. In this note we will not do this, but the interested reader preferring this expression might shift all the weights below by $3(\sigma + s)/8$.

3. Blowing up Γ

3.1. Since in the main construction we will need some of the operators induced by blowing down, we will make explicit the involved morphisms.

The next discussion provides a new proof of the stability of the lattice cohomology in the case when we blow up a vertex (for the old proof see [**N08**, (3.4)]).

Starting from now, the graph is neither necessarily connected, nor necessarily negative definite. Nevertheless, we will assume that the intersection form is non–degenerate.

We assume that Γ' is obtained from Γ by 'blowing up the vertex j_0'. More precisely, Γ' denotes a graph with one more vertex and one more edge: we glue to the vertex j_0 by the new edge the new vertex E_{new} with decoration -1 (and genus 0), while the decoration of E_{j_0} is modified from e_{j_0} into $e_{j_0} - 1$, and we keep all the other decorations. We will use the notations $L(\Gamma)$, $L(\Gamma')$, $L'(\Gamma)$, $L'(\Gamma')$. Let $w' : Char(\Gamma') \to \mathbb{Q}$ be the weight function of Γ' defined similarly as in (2.3.1). (We may use the following convention for the ordering of the indices: if $j \neq j_0$, then $j < j_0 < j_{new}$.) The following facts can be verified:

3.1.1. Consider the maps $\pi_* : L(\Gamma') \to L(\Gamma)$ defined by $\pi_*(\sum x_j E_j + x_{new}E_{new}) = \sum x_j E_j$, and $\pi^* : L(\Gamma) \to L(\Gamma')$ defined by $\pi^*(\sum x_j E_j) = \sum x_j E_j + x_{j_0}E_{new}$. Then $(\pi^* x, x')_{\Gamma'} = (x, \pi_* x')_\Gamma$. This shows that $(\pi^* x, \pi^* y)_{\Gamma'} = (x, y)_\Gamma$ and $(\pi^* x, E_{new})_{\Gamma'} = 0$ for any $x, y \in L(\Gamma)$. Both π_* and π^* extend over $L \otimes \mathbb{Q}$ and L'.

3.1.2. Set the (nonlinear) map: $c : L'(\Gamma) \to L'(\Gamma')$, $c(l') := \pi^*(l') + E_{new}$. Then $c(Char(\Gamma)) \subset Char(\Gamma')$ and c induces an isomorphism between the orbit spaces $Char(\Gamma)/2L(\Gamma) \simeq Char(\Gamma')/2L(\Gamma')$. Moreover, for any $k \in Char(\Gamma)$ and $k' := c(k)$ one has $w'(k') = w(k)$.

3.1.3. $\pi_*(Char(\Gamma')) \subset Char(\Gamma)$. For any $k' \in Char(\Gamma')$ write $k := \pi_*(k')$. Then $k' = \pi^*(k) + aE_{new}$ for some odd integer a, and $w'(k') - w(k) = \frac{a^2-1}{8} \in \mathbb{Z}_{\geq 0}$.

The maps c and π_* can be extended to the level of cubes and complexes as follows.

3.1.4. For any $q\geq 0$ and $\square=(k',I)\in\mathcal{Q}_q(\Gamma')$ one defines $\pi_*((k',I)):=(\pi_*(k'),I)\in\mathcal{Q}_q(\Gamma)$, provided that $j_{new}\notin I$. By (3.1.3) one has $w_{\Gamma'}(\square)-w_\Gamma(\pi_*(\square))\geq 0$. This defines a homological morphism $\pi_*^h:\mathcal{F}_q(\Gamma')\to\mathcal{F}_q(\Gamma)$ by

$$\pi_*^h(\square)=\begin{cases} U^{w_{\Gamma'}(\square)-w_\Gamma(\pi_*(\square))}\,\pi_*(\square) & \text{if } j_{new}\notin I,\\ 0 & \text{else.}\end{cases} \tag{3.1.5}$$

Using (2.3.3) one verifies that $\pi_*^h\circ\partial=\partial\circ\pi_*^h$, hence π_*^h is morphism of homological complexes. In particular, its dual $\pi_*^c:\mathcal{F}^q(\Gamma)\to\mathcal{F}^q(\Gamma')$, defined by $\pi_*^c(\phi)=\phi\circ\pi_*^h$, satisfies $\pi_*^c\circ\delta=\delta\circ\pi_*^c$, hence it is a morphism of complexes as well.

3.1.6. Before we extend c to the level of complexes, notice that for any $k\in Char(\Gamma)$

$$c(k+2E_j)=\begin{cases} c(k)+2E_j & \text{if } j\neq j_0,\\ c(k)+2E_j+2E_{new} & \text{if } j=j_0.\end{cases}$$

In particular, the pair $c(k)$ and $c(k+2E_{j_0})$ does not form a 1–cube in Γ'. Hence, the application $(k,I)\mapsto(c(k),I)$, sending the vertex $k+2\sum_{j\in I'}E_j$ into $c(k)+2\sum_{j\in I'}E_j$, *does not* commute with the boundary operator. The 'right' operator $c^h:\mathcal{F}_q(\Gamma)\to\mathcal{F}_q(\Gamma')$ is defined by

$$c^h((k,I))=\begin{cases} (c(k),I) & \text{if } j_0\notin I,\\ (c(k),I)+U^{w(k,I)-w(k+2E_{j_0},I_0)}(c(k)+2E_{j_0},I_0\cup j_{new}) & \text{if } I=I_0\cup j_0,\ \ j_0\notin I_0.\end{cases} \tag{3.1.7}$$

In fact, by (3.1.8) below, $w(k+2E_{j_0},I_0)$ above can be replaced by $w'(c(k)+2E_{j_0},I_0)$, or even by $w'(c(k)+2E_{j_0},I_0\cup j_{new})$. By (3.1.2) and a computation one gets (where in the third line I_0' is any subset of $\mathcal{J}$ with $j_0\notin I_0'$):

$$\begin{aligned} w((k,I)) &= w'((c(k),I)),\\ w((k+2E_{j_0},I_0)) &= w'((c(k)+2E_{j_0},I_0)),\\ w'(c(k)+2\textstyle\sum_{j\in I_0'}E_j+2E_{j_0}) &= w'(c(k)+2\textstyle\sum_{j\in I_0'}E_j+2E_{j_0}+2E_{new}).\end{aligned} \tag{3.1.8}$$

These and a (longer) computations shows that c^h commutes with the boundary operator ∂.

3.1.9. Using $\pi_*\circ c$, the definitions and the first equation of (3.1.8) one gets $\pi_*^h\circ c^h=id_{\mathcal{F}_*(\Gamma)}$. On the other hand, $c^h\circ\pi_*^h$ is not the identity, but it is homotopic to $id_{\mathcal{F}_*(\Gamma')}$. Indeed, we define the homotopy operator $K:\mathcal{F}_*(\Gamma')\to\mathcal{F}_{*+1}(\Gamma')$ as follows. Write any k as $c\pi_*(k)+2aE_{new}$ for some $a\in\mathbb{Z}$. Then define $K((k,I))$ as 0 if either $j_{new}\in I$ or $a=0$. Otherwise take for $K((k,I))$:

$$sign(a)\cdot\sum U^{w(k,I)-w(c\pi_*(k)+2lE_{new},I\cup j_{new})}\,(c\pi_*(k)+2lE_{new},I\cup j_{new}),$$

where the summation is over $l\in\{0,1,\dots,a-1\}$ if $a>0$ and $l\in\{a,\dots,-1\}$ if $a<0$. (The exponents are non–negative because of (3.1.3).) Then, again by a computation, $\partial\circ K-K\circ\partial=id-c^h\circ\pi_*^h$.

In particular, π_*^h and c^h induce (degree preserving) isomorphisms of the corresponding lattice cohomologies. In the sequel the operator π_*^h will be crucial.

4. Comparing Γ and $\Gamma\setminus j_0$

4.1. Notations, remarks. We consider a non–degenerate graph Γ as in 3.1, and we fix one of its vertices $j_0\in\mathcal{J}$. The new graph $\Gamma\setminus j_0$ is obtained from Γ by deleting the vertex j_0 and all its adgacent edges. We will denote by $L(\Gamma),L'(\Gamma),L(\Gamma\setminus j_0),L'(\Gamma\setminus j_0)$ the corresponding lattices.

The operator $i:L(\Gamma\setminus j_0)\to L(\Gamma)$, $i(E_{j,\Gamma\setminus j_0})=E_{j,\Gamma}$ identifies $L(\Gamma\setminus j_0)$ with a sublattice of $L(\Gamma)$. The dual operator (restriction) is $R:L'(\Gamma)\to L'(\Gamma\setminus j_0)$, $R(E^*_{j,\Gamma})=E^*_{j,\Gamma\setminus j_0}$ for $j\neq j_0$ and $R(E^*_{j_0,\Gamma})=0$. It satisfies $(i_\mathbb{Q}(x),y)_\Gamma=(x,R(y))_{\Gamma\setminus j_0}$ for any $x\in L'(\Gamma\setminus j_0)$ and $y\in L'(\Gamma)$. (Here, $E^*_{j,\Gamma}$ respectively $E^*_{j,\Gamma\setminus j_0}$ are the usual dual generators of L' considered in Γ, respectively in $\Gamma\setminus j_0$.)

Recall that $l'=\sum_j a_jE^*_{j,\Gamma}$ is characteristic if and only if $a_j\equiv e_j\ (mod\,2)$. In particular, R sends characteristic elements into characteristic elements. On the other hand, $i_\mathbb{Q}$ not necessarily preserves characteristic elements.

Although $R(Char(\Gamma))\subset Char(\Gamma\setminus j_0)$, this operator cannot be extended to the level of cubes. Indeed, notice that

$$R(E_{j_0})=-\sum_{(j,j_0)\in\mathcal{E}_\Gamma}E^*_{j,\Gamma\setminus j_0}, \tag{4.1.1}$$

where the sum runs over the adjacent vertices of j_0 in Γ. In particular, the endpoints of the 1–cube (k,j_0) are sent into $R(k)$ and $R(k)+2R(E_{j_0})$, which cannot be expressed as combination of 1–cubes in $\Gamma\setminus j_0$. In

the next subsection we will consider another operator, which can be extended to the level of cubes (and which, in fact, operates in a different direction than R, cf. 2.4).

4.2. The B–operator. Consider $b : L'(\Gamma\setminus j_0)\to L'(\Gamma)$ defined by

$$\sum_j a_j E^*_{j,\Gamma\setminus j_0}\mapsto \sum_j a_j E^*_{j,\Gamma}.$$

Clearly, if $k\in Char(\Gamma\setminus j_0)$ then $b(k)+a_{j_0}E^*_{j_0,\Gamma}\in Char(\Gamma)$ for any a_{j_0} with $a_{j_0}\equiv e_{j_0}$ $(mod\ 2)$. In order to see how it operates on cubes, notice that in $\Gamma\setminus j_0$ for any $j\in\mathcal{J}(\Gamma\setminus j_0)$ one has

$$E_{j,\Gamma\setminus j_0}=-e_j\,E^*_{j,\Gamma\setminus j_0}-\sum_{(i,j)\in\mathcal{E}_{\Gamma\setminus j_0}}E^*_{i,\Gamma\setminus j_0},$$

and a similar relation holds in Γ too. Therefore, one gets that

$$b(E_j)=E_j+(E_{j_0},E_j)_\Gamma\cdot E^*_{j_0,\Gamma}. \tag{4.2.1}$$

Therefore,

$$b(l')=i_{\mathbb{Q}}(l')+(E_{j_0},i_{\mathbb{Q}}(l'))_\Gamma\cdot E^*_{j_0,\Gamma}. \tag{4.2.2}$$

In particular, b is a 'small' modification of $i_{\mathbb{Q}}$, but this modification is enough to extend it to the level of cubes. Although the vertices of (k,I) are not sent to the vertices of a cube (provided that I contains elements adjacent to j_0 in Γ), cf. (4.2.1), nevertheless if we consider all the shifts in the direction of the 'error' of (4.2.1) we get a well–defined operator

$$B_q:\mathcal{F}_q(\Gamma\setminus j_0)\to\mathcal{F}_q(\Gamma)$$

given by

$$B_q((k,I)):=\sum_{a_{j_0}\equiv e_{j_0}\ (mod\ 2)}(b(k)+a_{j_0}E^*_{j_0,\Gamma},I). \tag{4.2.3}$$

(Here we keep the notation 'B', since this notation was used in similar contexts, as in [**OSz03, N05, Gr08**].)

B_q induces a morphism $B^q:\mathcal{F}^q(\Gamma)\to\mathcal{F}^q(\Gamma\setminus j_0)$ via $(B^q\phi)(\square)=\phi(B_q\square)$. A straightforward (slightly long) computation shows that B_q commutes with the boundary operator ∂, hence $B^*\circ\delta=\delta\circ B^*$ too. In particular, one gets a well–defined morphism of $\mathbb{Z}[U]$–modules $\mathbb{B}^*:\mathbb{H}^*(\Gamma)\to\mathbb{H}^*(\Gamma\setminus j_0)$.

4.3. The sign–modified B–operator. In the exact sequences considered in the next section we will need to modify the B–operator by a sign (compare with the end of §2 of [**OSz03**], where the case $q=0$ is discussed). The definition depends on some choices.

For each L–orbit $[k]:=k+2L(\Gamma\setminus j_0)\subset Char(\Gamma\setminus j_0)$ we will fix a representative $r_{[k]}\in[k]$. Hence, for any characteristic element k, with orbit $[k]$, $k-r_{[k]}\in 2L(\Gamma\setminus j_0)$, hence $(k-r_{[k]},E^*_{j,\Gamma\setminus j_0})\in 2\mathbb{Z}$ for any $j\neq j_0$. In particular, $(k-r_{[k]},R(E_{j_0}))\in 2\mathbb{Z}$ for $R(E_{j_0})$ defined in (4.1.1). Then the modified operator

$$\overline{B}_q:\mathcal{F}_q(\Gamma\setminus j_0)\to\mathcal{F}_q(\Gamma)$$

is given by

$$\overline{B}_q((k,I)):=\sum_{a_{j_0}\equiv e_{j_0}\ (mod\ 2)}(-1)^{n(k,a_{j_0})/2}\cdot(b(k)+a_{j_0}E^*_{j_0,\Gamma},I), \tag{4.3.1}$$

where

$$n(k,a_{j_0}):=(k-r_{[k]},R(E_{j_0}))+a_{j_0}+e_{j_0}.$$

Then, again, $\overline{B}^*$ commutes with the boundary operator, hence $\overline{B}^*:\mathcal{F}^q(\Gamma)\to\mathcal{F}^q(\Gamma\setminus j_0)$ defined by $(\overline{B}^q\phi)(\square)=\phi(\overline{B}_q\square)$ commutes with δ, hence it also defines a $\mathbb{Z}[U]$–modules morphism $\overline{\mathbb{B}}^*:\mathbb{H}^*(\Gamma)\to\mathbb{H}^*(\Gamma\setminus j_0)$.

Remark 4.3.2. The morphisms $\mathbb{B}^*$ and $\overline{\mathbb{B}}^*$ do *not* preserve the gradings *neither* the orbits $Char/2L$ (i.e. they do not split in direct sum with respect to these orbits).

5. The surgery exact sequence

5.1. Notations. Let Γ be a non–degenerate graph as above with $s=|\mathcal{J}|\geq 2$. We fix a vertex $j_0\in\mathcal{J}$. Associated with j_0 we consider two other graphs, namely $\Gamma\setminus j_0$ and $\Gamma^+_{j_0}$. The second one is obtained from Γ by modifying the decoration e_{j_0} of the vertex j_0 into $e_{j_0}+1$. In order to stay in our category of objects, we will assume that both graphs $\Gamma\setminus j_0$ and $\Gamma^+_{j_0}$ are non–degenerate.

Our goal is to establish a long exact sequence connecting the lattice cohomologies of these three graphs, similar which is valid for the Heegaard–Floer cohomologies provided by surgeries. As usual, in order to define a long exact sequence, we need first to determine a short exact sequence of complexes.

The graphs Γ and $\Gamma\setminus j_0$ will be connected by the $\overline{B}$–operator, cf. 4.3. We define the 'A'–operator connecting Γ and $\Gamma^+_{j_0}$ as follows. Let Γ^b be the graph obtained from Γ by attaching a new vertex j_{new} (via a single new edge) to j_0, such that the decoration of the new vertex is -1. Then, for the pair Γ, Γ^b (since $\Gamma=\Gamma^b\setminus j_{new}$), we can apply the construction and results of §4. In particular, we get morphisms of complexes: $B_*:\mathcal{F}_*(\Gamma)\to\mathcal{F}_*(\Gamma^b)$ and $B^*:\mathcal{F}^*(\Gamma^b)\to\mathcal{F}^*(\Gamma)$. Next, since $\Gamma^+_{j_0}$ obtained from Γ^b by blowing down the new vertex j_{new}, by § 3 we get morphisms of complexes: $\pi^h_*:\mathcal{F}_*(\Gamma^b)\to\mathcal{F}_*(\Gamma^+_{j_0})$ and $\pi^c_*:\mathcal{F}^*(\Gamma^+_{j_0})\to\mathcal{F}^*(\Gamma^b)$. By composition we get the A–operators:

$$A_*:=\pi^h_*\circ B_*:\mathcal{F}_*(\Gamma)\to\mathcal{F}_*(\Gamma^+_{j_0}),\ \text{ and }\ A^*:=B^*\circ\pi^c_*:\mathcal{F}^*(\Gamma^+_{j_0})\to\mathcal{F}^*(\Gamma).$$

In particular, we can consider the short sequence of complexes

$$0\to\mathcal{F}^*(\Gamma^+_{j_0})\xrightarrow{A^*}\mathcal{F}^*(\Gamma)\xrightarrow{\overline{B}^*}\mathcal{F}^*(\Gamma\setminus j_0)\to 0. \tag{5.1.1}$$

Theorem 5.1.2. *The short sequence of complexes (5.1.1) is exact.*

The proof is given in several steps.

5.2. The injectivity of $\mathbf{A^q}$. Take an arbitrary non–zero $\phi\in\mathcal{F}^q(\Gamma^+_{j_0})$. In order to prove that $A^q\phi\neq 0$, we need to find $f\in\mathcal{F}_q(\Gamma)$ such that $\phi(A_q(f))\neq 0$. Let N be the smallest non–negative integer such that $U^{N+1}\phi=0$. Then replacing ϕ by $U^N\phi$ we may assume that $U\phi=0$. In particular, in f (or in $A_q(f)$) any term whose coefficient has the form U^n $(n>0)$ is irrelevant.

Since $\phi\neq 0$, there exists $(\bar{k},I)\in\mathcal{Q}_q(\Gamma^+_{j_0})$ with $\phi((\bar{k},I))\neq 0$. Write

$$\bar{k}=\sum_{j\in\mathcal{J}}a_jE^*_{j,\Gamma^+_{j_0}}+E^*_{j_0,\Gamma^+_{j_0}}, \tag{5.2.1}$$

with a_j-e_j even for all $j\in\mathcal{J}$. Since ϕ is finitely supported, we may assume that

$$a_{j_0}\ \text{is minimal with the property}\ \phi((\bar{k},I))\neq 0. \tag{5.2.2}$$

We wish to construct $f\in\mathcal{F}_q(\Gamma)$ such that $\phi(A_q(f)-(\bar{k},I))=0$. For the weight functions in Γ^b and $\Gamma^+_{j_0}$ we will use the notations w^b respectively w^+.

First, set

$$k:=\sum_j a_jE^*_{j,\Gamma}\in Char(\Gamma). \tag{5.2.3}$$

Then

$$B_0(k)=\sum_{a\equiv 1}k^b_a,\quad\text{where}\quad k^b_a=\sum_j a_jE^*_{j,\Gamma^b}+aE^*_{new}.$$

Set also

$$k_a:=\pi_*(k^b_a)=\sum_j a_jE^*_{j,\Gamma^+_{j_0}}+aE^*_{j_0,\Gamma^+_{j_0}}.$$

Notice that $k_1=\bar{k}$. Since $\pi^*k_a=k^b_a-aE_{new}$, by (3.1.3)

$$w^b(k^b_a)=w^+(k_a)+(a^2-1)/8. \tag{5.2.4}$$

Moreover, $B((k,I))=\sum_{a\equiv 1}(k^b_a,I)$ and $\pi_*((k^b_a,I))=(k_a,I)$.

For any $I'\subset I$ write $E_{I'}=\sum_{j\in I'}E_j$. Let $\delta_{I'}$ be 1 if $j_0\in I'$, and zero otherwise. Since

$$w^b(k^b_a+2E_{I'})-w^b(k^b_a)=\frac{1}{2}\sum_{j\in I'}a_j-\frac{1}{2}(E_{I'},E_{I'})_{\Gamma^b}, \tag{5.2.5}$$

and

$$w^+(k_a+2E_{I'})-w^+(k_a)=\frac{1}{2}\sum_{j\in I'}a_j+\frac{1}{2}\delta_{I'}a-\frac{1}{2}(E_{I'},E_{I'})_{\Gamma^+_{j_0}}, \tag{5.2.6}$$

we get

$$w^b(k^b_a+2E_{I'})-w^+(k_a+2E_{I'})=\frac{a^2-1}{8}-\delta_{I'}\cdot\frac{a-1}{2}. \tag{5.2.7}$$

Assume that $I\not\ni j_0$. Then, by (5.2.7), we get that $w^b(k^b_a,I)-w^+(k_a,I)=(a^2-1)/8$, hence

$$A_q((k,I))=\sum_{a\equiv 1,\,a<0}U^{\frac{a^2-1}{8}}(k_a,I)+(\bar{k},I)+\sum_{a\equiv 1,\,a\geq 3}U^{\frac{a^2-1}{8}}(k_a,I). \tag{5.2.8}$$

Both sums are killed by ϕ, the first one by (5.2.2), the second one by $U\phi=0$, hence $\phi(A_q((k,I))-(\bar{k},I))=0$.

Next, assume that $I\ni j_0$. In that case, again by (5.2.7), we get that $w^b(k^b_a,I)-w^+(k_a,I)>0$ whenever $a\notin\{-1,1,3\}$. For the other three special values we have the following facts. It is convenient to set for each k (represented as in (5.2.3)) the integer:

$$M(k):=\max_{I'}\{\sum_{j\in I'}a_j-(E_{I'},E_{I'})_{\Gamma^b}\}. \tag{5.2.9}$$

- For $a=1$ one has $w^b(k^b_a,I)-w^+(k_a,I)=0$ always.
- If $a=3$, then the right hand side of (5.2.7) is $1-\delta_{I'}$, hence $w^b(k^b_a,I)-w^+(k_a,I)=0$ if and only if $\max_{I'}\{w^b(k^b_a+2E_{I'})\}$ can be realized by some I' with $I'\ni j_0$. By (5.2.5) this is equivalent to

$$M(k)\text{ can be realized by some } I' \text{ with } I'\ni j_0. \tag{5.2.10}$$

- If $a=-1$, then the right hand side of (5.2.7) is $\delta_{I'}$, hence $w^b(k^b_a,I)-w^+(k_a,I)=0$ if and only if

$$M(k)\text{ can be realized by some } I' \text{ with } I'\not\ni j_0. \tag{5.2.11}$$

Assume that in the case $I\ni j_0$ for $a=3$ one has $w^b(k^b_a,I)-w^+(k_a,I)>0$. Then by an identical argument as in the case $I\not\ni j_0$ we get that $\phi(A_q((k,I))-(\bar{k},I))=0$.

Hence, the remaining final case is when there exists at least one $I'\subset I$ which contains j_0 and satisfies (5.2.10). Then

$$A_q((k,I)=(\bar{k},I)+(\bar{k}+2E^*_{j_0,\Gamma^+_{j_0}},I)\quad (mod\ U \text{ and } \ker(\phi)).$$

Now, we will consider $k+2E^*_{j_0,\Gamma}$ instead of k. Via the operator A_q it provides the same cubes as k (where the index set will have a shift $a\mapsto a-2$) but with different U^m–coefficients. Notice that by (5.2.10) and (5.2.11), $M(k+2E^*_{j_0,\Gamma})$ can be realized only by subsets I' with $I'\ni j_0$, hence (5.2.11) will fail. Therefore,

$$A_q((k+2E^*_{j_0,\Gamma},I)=(\bar{k}+2E^*_{j_0,\Gamma^+_{j_0}},I)+(\bar{k}+4E^*_{j_0,\Gamma^+_{j_0}},I)\quad (mod\ U).$$

More generally, for any positive integer ℓ, by the same argument:

$$A_q((k+2\ell E^*_{j_0,\Gamma},I)=(\bar{k}+2\ell E^*_{j_0,\Gamma^+_{j_0}},I)+(\bar{k}+(2\ell+2)E^*_{j_0,\Gamma^+_{j_0}},I)\quad (mod\ U).$$

Since ϕ is finitely supported, $\phi((2\ell+2)E^*_{j_0,\Gamma^+_{j_0}},I))=0$ for some $\ell\geq 0$, let us consider the minimal such ℓ. Then A_q, modulo U and $\ker(\phi)$, restricted on the relevant finite dimensional spaces looks as an $(\ell+1)\times(\ell+1)$ upper triangular matrix whose diagonal is the identity matrix. Since this is invertible over $\mathbb{Z}$, the result follows: some linear combinations of the elements $A_q((k+2tE^*_{j_0,\Gamma},I)$, $0\leq t\leq\ell$, is $(\bar{k},I)$ modulo U and $\ker(\phi)$.

5.3. The surjectivity of $\overline{\mathbf{B}}^{\mathbf{q}}$. We provide the same argument as [**Gr08**]: For any fixed $a\equiv e_{j_0}\ (mod\,2)$,

$$\overline{B}^q\big(U^{-\ell}*(b(k)+aE^*_{j_0,\Gamma},I)^\vee\big)=\pm U^{-\ell}*(k,I)^\vee.$$

Since the collection of $U^{-\ell}*(k,I)^\vee$ generate $\mathcal{F}^q(\Gamma\setminus j_0)$ (over $\mathbb{Z}$), the surjectivity follows.

5.4. $\overline{\mathbf{B}}^{\mathbf{q}}\circ\mathbf{A}^{\mathbf{q}}=\mathbf{0}$. Take an arbitrary $(k,I)\in\mathcal{Q}_q(\Gamma\setminus j_0)$. Then, by (5.2.7), one has:

$$(A_q\circ\overline{B}_q)((k,I))=\sum_{a\equiv 1}\ \sum_{c\equiv e_{j_0}}(-1)^{N(k)+\frac{c+e_{j_0}}{2}}\cdot U^{\frac{a^2-1}{8}}\big(\pi_*b(k)+(a+c)E^*_{j_0,\Gamma^+_{j_0}},I\big).$$

Those pairs (a,c) for which a^2-1 and $a+c$ are fixed hit the same element of $\mathcal{F}_q$. Write $a=2i+1$. Then the two solutions of $(a^2-1)/8=i(i+1)/2=t$ satisfies $i_1+i_2=-1$. Since the corresponding two c values satisfies $2i_1+c_1=2i_2+c_2$, one gets that $(c_2-c_1)/2$ is odd. Hence the terms cancel each other two by two.

5.5. $\ker \overline{\mathbf{B}}^{\mathbf{q}} \subset \mathbf{im}\,\mathbf{A}^{\mathbf{q}}$**.** Set $\ker U^m := \{\phi \in \mathcal{F}^q(\Gamma)\,:\, U^m\phi = 0\}$. Notice that the inclusion

$$\ker U^m \cap \ker \overline{B}^q \subset im\,A^q \tag{5.5.1}$$

for $m = 1$ (together with (5.4)) implies by induction its validity for any m (cf. [**Gr08**]). Indeed, assume that the inclusion is true for $m-1$ and set $\phi \in \ker U^m \cap \ker \overline{B}^q$. Then $U\phi = A^q(\psi)$ for some ψ. Moreover, $\tilde{\phi} := \phi - A^q(U^{-1} * \psi) \in \ker U$. On the other hand, by (5.4), $\tilde{\phi} \in \ker \overline{B}^q$ too. Therefore, by (5.5.1) applied for $m = 1$, we get $\tilde{\phi} \in im\,A^q$, hence $\phi \in im\,A^q$ too.

Notice that $\cup_m(\ker U^m \cap \ker \overline{B}^q) = \ker \overline{B}^q$, hence this would prove $\ker \overline{B}^q \subset im\,A^q$ too.

Next, we show (5.5.1) for $m = 1$. First notice that $\ker U \cap \ker \overline{B}^q$ is generated by elements of type $(k, I)^\vee + (k + 2E^*_{j_0,\Gamma}, I)^\vee$ where $I \not\ni j_0$, and $(k, I)^\vee$ where $I \ni j_0$.

In the first case (i.e. $I \not\ni j_0$), let us write k as in (5.2.3), and set $\bar{k}$ as in (5.2.1). Then by (5.2.7) and (5.2.8) one has

$$A_q((k, I)) = (\bar{k} - 2E^*_{j_0,\Gamma^+_{j_0}}, I) + (\bar{k}, I) \quad (mod\ U),$$

$$A_q((k + 2E^*_{j_0,\Gamma}, I)) = (\bar{k}, I) + (\bar{k} + 2E^*_{j_0,\Gamma^+_{j_0}}, I) \quad (mod\ U),$$

and $(\bar{k}, I)$ does not appear in any other term with non–zero coefficient $(mod\ U)$. Hence $A^q(\bar{k}, I)^\vee = (k, I)^\vee + (k + 2E^*_{j_0,\Gamma}, I)^\vee$.

Next, we fix an element of type $(k, I)^\vee$ with $I \ni j_0$. It belongs to the collection $\{(k(i), I)\}_{a\in\mathbb{Z}}$, where $k(i) = k + 2iE^*_{j_0,\Gamma}$, which will be treated simultaneously via the discussions of (5.2). Write k as in (5.2.3) and set $\bar{k}$ via (5.2.1); it is also convenient to write $\bar{k}(i) := \bar{k} + 2iE^*_{j_0,\Gamma^+_{j_0}}$. Notice that $k(i)$ has coefficients $\{\{a_j\}_{j\neq j_0}, a_{j_0} + 2i\}$. Therefore, for $i \ll 0$ (5.2.10) will fail and (5.2.11) is satisfied. Let $i_0 - 1$ be maximal when (5.2.10) fails. Then for $i = i_0$ both conditions are satisfied, and for $i > i_0$ only (5.2.10). Therefore,

$$\begin{array}{lll} A_q((k(i), I)) = (\bar{k}(i-1), I) + (\bar{k}(i), I) & (mod\ U) & \text{if } i < i_0, \\ A_q((k(i_0), I)) = (\bar{k}(i_0 - 1), I) + (\bar{k}(i_0), I) + (\bar{k}(i_0 + 1), I) & (mod\ U) & \\ A_q((k(i), I)) = (\bar{k}(i), I) + (\bar{k}(i+1), I) & (mod\ U) & \text{if } i > i_0. \end{array} \tag{5.5.2}$$

This reads as

$$A^q((\bar{k}(i), I)^\vee) = \begin{cases} (k(i), I)^\vee + (k(i+1), I)^\vee & \text{if } i < i_0 \\ (k(i), I)^\vee & \text{if } i = i_0 \\ (k(i), I)^\vee + (k(i-1), I)^\vee & \text{if } i > i_0 \end{cases}$$

Taking finite linear combination we get that any $(k(i), I)^\vee$ is in the image of A^q. This ends the proof of Theorem (5.1.2). As a corollary we get:

Theorem 5.5.3. *Assume that the graphs $\Gamma^+_{j_0}$, Γ and $\Gamma \setminus j_0$ are non–degenerate. Then*

$$\cdots \longrightarrow \mathbb{H}^q(\Gamma^+_{j_0}) \xrightarrow{\mathbb{A}^q} \mathbb{H}^q(\Gamma) \xrightarrow{\overline{\mathbb{B}}^q} \mathbb{H}^q(\Gamma \setminus j_0) \xrightarrow{\mathbb{C}^q} \mathbb{H}^{q+1}(\Gamma^+_{j_0}) \longrightarrow \cdots$$

is an exact sequence of $\mathbb{Z}[U]$–modules.

6. The exact sequence for negative definite graphs

6.1. Preliminaries. For any graph Γ we write $\det(\Gamma)$ for the determinant of the *negative* of the intersection form associated with Γ. If Γ is negative definite then $\det(\Gamma)$ is obviously positive.

If Γ is negative definite then $\Gamma \setminus j_0$ is automatically so for any j_0. Nevertheless, this is not the case for $\Gamma^+_{j_0}$ (although, if $\Gamma^+_{j_0}$ is negative definite then Γ is so too).

Lemma 6.1.1. *Assume that Γ is negative definite. Then $\Gamma^+_{j_0}$ is negative definite if and only if $\det(\Gamma) > \det(\Gamma \setminus j_0)$. If these conditions are satisfied then $E^*_{j_0,\Gamma} \notin L(\Gamma)$ and $(E^*_{j_0,\Gamma})^2 \notin \mathbb{Z}$.*

Proof. $\Gamma^+_{j_0}$ is negative definite if and only if $\det(\Gamma^+_{j_0})$ is positive (provided that $\Gamma \setminus j_0$ is negative definite). But $\det(\Gamma^+_{j_0}) = \det(\Gamma) - \det(\Gamma \setminus j_0)$. The last statement follows from $-(E^*_{j_0,\Gamma})^2 = \det(\Gamma \setminus j_0)/\det(\Gamma)$, which cannot be an integer. □

In the sequel we assume that all the graphs are negative definite, but not necessarily connected. The next theorem is an addendum of Theorem 5.5.3. Since its proof is rather long, it will be published elsewhere.

Theorem 6.1.2. *Assume that $\Gamma^+_{j_0}$ is negative definite. Consider the exact sequence*

$$0 \longrightarrow \mathbb{H}^0(\Gamma^+_{j_0}) \xrightarrow{\mathbb{A}^0} \mathbb{H}^0(\Gamma) \xrightarrow{\overline{\mathbb{B}}^0} \mathbb{H}^0(\Gamma \setminus j_0) \xrightarrow{\mathbb{C}^0} \mathbb{H}^1(\Gamma^+_{j_0}) \longrightarrow \cdots$$

and the canonical submodule $\mathbb{T}(\Gamma \setminus j_0)$ of $\mathbb{H}^0(\Gamma \setminus j_0)$. Then $\mathbb{T}(\Gamma \setminus j_0) \subset im\,\overline{\mathbb{B}}^0$; or, equivalently, the restriction $\mathbb{C}^0|\mathbb{T}(\Gamma \setminus j_0)$ is zero.

6.2. Graphs with n bad vertices. We say that a negative definite connected graph is 'rational' if it is the plumbing representation of a the link of a rational singularity (or, the resolution graph of a rational singularity). They were characterized combinatorially by Artin, for more details see [**N99, N05**].

We fix an integer $n \geq 0$. We say that a negative definite graph has at most n 'bad' vertices if we can find n vertices $\{j_k\}_{1\leq k\leq n}$, such that replacing their decorations e_{j_k} by some more negative integers $e'_{j_k} \leq e_{j_0}$ we get a graph whose all connected components are rational. (Notice that this is a generalization of the notion of 'bad' vertices of [**OSz03**]. A graph with at most one bad vertex is called 'almost rational' in [**N05, N08**]. Any 'star–shaped' graph, i.e. normal form of a Seifert manifold, has at most one bed vertex, namely the 'central' vertex.)

Theorem 6.2.1. *If Γ has at most n bad vertices then $\mathbb{H}^q_{red}(\Gamma) = 0$ for $q \geq n$.*

This is a generalization of [**N08**, (4.3.3)], where it is proved for $n = 1$ (compare also with the vanishing theorems of [**OSz03, N05**]).

Proof. We run induction over n. If $n = 0$, then all the components of Γ are rational. By [**N05**], their reduced lattice cohomology is vanishing. This fact remains true for more components too, since the cohomology of a tensor product of two acyclic complexes is acyclic.

Assume now that the statement is true for $n - 1$ and take Γ with n bad vertices. Let j be one of them. Let $\Gamma_j(-\ell)$ be the graph obtained from Γ by replacing the decoration e_j by $e_j - \ell$ ($\ell \geq 0$). Then consider the long exact sequence (5.5.3) associated with $\Gamma_j(-\ell)$, $\Gamma_j(-\ell - 1)$ and $\Gamma \setminus j_0$, for all $\ell \geq 0$. Then, by the inductive step, we get that $\mathbb{H}^q(\Gamma) = \mathbb{H}^q(\Gamma_j(-\ell))$ for all ℓ and $q \geq n$. (Here, in the case $n = 1$, Theorem 6.1.2 is also used.) Since for $\ell \gg 0$ the graph $\Gamma_j(-\ell)$ has only $n - 1$ bad vertices, all these modules vanish. □

In fact, the above statement can be improved as follows.

Theorem 6.2.2. *Assume that Γ has at most $n \geq 2$ bad vertices $\{j_k\}_{1\leq k\leq n}$ such that $\Gamma \setminus j_1$ has at most $(n - 2)$ bad vertices. Then $\mathbb{H}^q_{red}(\Gamma) = 0$ for $q \geq n - 1$.*

Proof. The proof is same as above, if one eliminates first the vertex j_1. □

See [**N05**, (8.2)(5.b)] for a graph Γ with 2 bad vertices $\{j_1, j_2\}$ such that $\Gamma \setminus j_1$ has only rational components.

7. The 'relative' surgery exact sequence

7.1. Preliminaries. The motivation for the next exact sequence is two–folded. First, the exact sequence (5.5.3) mixes the classes of the characteristic elements. (Note that the surgery exact sequence valid for the Heegaard–Floer theory — which is one of out models for the theory — does the same.) These classes, in topological language, correspond to the *spin*^{c}–structures of the corresponding plumbed 3–manifolds. It would be desirable to have a surgery exact sequence which do not mix them, and allows inductively the computation of each $\mathbb{H}^*(\Gamma, [k])$ for each $[k]$ independently.

The second motivation is the main result of [**BN10**]. This is a surgery formula for the Seiberg–Witten invariant of negative definite plumbed 3–manifolds, it compares these invariants for Γ and $\Gamma \setminus j_0$ for fixed (non–mixed) *spin*^{c}–structures. The third term in the main formula of [**BN10**] comes from a 'topological' Poincaré series associated with the plumbing graph, and its nature is rather different than the other two terms.

Here our goal is to determine an exact sequence connecting $\mathbb{H}^*(\Gamma, [\bar{k}])$ and $\mathbb{H}^*(\Gamma \setminus j_0, [R(\bar{k})])$ (where $R(\bar{k})$ is the restriction of $\bar{k}$, see 4.1) with the newly defined third term, playing the role of the relative cohomology. Its relationship with the Poincaré series used in [**BN10**] will also be treated.

7.2. The 'relative' complex and cohomology. We consider a non–degenerate graph Γ and j_0 one of its vertices.

We fix $[k] \in Char(\Gamma \setminus j_0)/2L(\Gamma \setminus j_0)$ and a characteristic element $k_m \in [k]$ with $w(k_m) = \min_{k\in[k]} w(k)$. Furthermore, we fix a_0 satisfying $a_0 \equiv e_{j_0} \pmod 2$. Then $k_{a_0} := i(k_m) + \big((i(k_m), E_{j_0}) + a_0 \big) E^*_{j_0} \in Char(\Gamma)$. In fact, for any $k' \in [k]$ one gets that $i(k') + \big((i(k_m), E_{j_0}) + a_0 \big) E^*_{j_0} \in Char(\Gamma)$ and it is an element of $[k_{a_0}]$. For simplicity we write r_0 for $(i(k_m), E_{j_0}) + a_0$.

We define

$$B_{0,rel} : \mathcal{F}_0(\Gamma \setminus j_0, [k]) \to \mathcal{F}_0(\Gamma, [k_{a_0}])$$

by

$$B_{0,rel}(k') = i(k') + \big((i(k_m), E_{j_0}) + a_0 \big) E^*_{j_0} = i(k') + r_0 E^*_{j_0}. \tag{7.2.1}$$

This extends to the level of complexes $B_{*,rel}:(\mathcal{F}_*(\Gamma\setminus j_0,[k]),\partial)\to(\mathcal{F}_*(\Gamma,[k_{a_0}]),\partial)$ by $B_{*,rel}((k,I))=(B_{0,rel}(k),I)$. Its dual $B^*_{rel}:(\mathcal{F}^*(\Gamma,[k_{a_0}]),\delta)\to(\mathcal{F}^*(\Gamma\setminus j_0,[k]),\delta)$ is defined by $B^*_{rel}(\phi)=\phi\circ B_{*,rel}$.

By a similar argument as in (5.3) we get that

$$B^*_{rel}:\mathcal{F}^*(\Gamma,[k_{a_0}])\to\mathcal{F}^*(\Gamma\setminus j_0,[k])\quad\text{is surjective.}$$

We define the *'relative'* complex $\mathcal{F}^*_{rel}=\mathcal{F}^*_{rel}(\Gamma,j_0,[k],a_0])$ via $\ker(B^*_{rel})$. Let the cohomology of the complex $(\mathcal{F}^*_{rel},\delta)$ be $\mathbb{H}^*_{rel}=\mathbb{H}^*_{rel}(\Gamma,j_0,[k],a_0)$. It is a graded $\mathbb{Z}[U]$–module. We refer to it as the *relative lattice cohomology.*

Note that both $\mathcal{F}^*_{rel}$ and B^*_{rel} depend on the choice of the representative k_{a_0} of $[k_{a_0}]$ and are *not* invariants merely of the classes $[k_{a_0}]$ and $[k]$.

Theorem 7.2.2. *One has the short exact sequence of complexes:*

$$0\longrightarrow\mathcal{F}^*_{rel}\xrightarrow{A^*_{rel}}\mathcal{F}^*(\Gamma,[k_{a_0}])\xrightarrow{B^*_{rel}}\mathcal{F}^*(\Gamma\setminus j_0,[k])\longrightarrow 0,$$

which provides a long exact sequence of $\mathbb{Z}[U]$–modules:

$$\cdots\longrightarrow\mathbb{H}^q_{rel}\xrightarrow{\mathbb{A}^q_{rel}}\mathbb{H}^q(\Gamma,[k_{a_0}])\xrightarrow{\mathbb{B}^q_{rel}}\mathbb{H}^q(\Gamma\setminus j_0,[k])\xrightarrow{\mathbb{C}^q_{rel}}\mathbb{H}^{q+1}_{rel}\longrightarrow\cdots$$

(Again, the relative cohomology modules and the operators in the above exact sequence depend on the choice of k_{a_0}, and not only on its class $[k_{a_0}]$.)

In the case when Γ is negative definite, the restriction of $\mathbb{B}^0_{rel}$ to $\mathcal{T}^+_{d[k_{a_0}]}$ has its image in $\mathcal{T}^+_{d[k]}$.

Proposition 7.2.3. *Assume that Γ (but not necessarily $\Gamma^+_{j_0}$) is negative definite. Then*

$$\mathbb{B}^0_{rel}:\mathcal{T}^+_{d[k_{a_0}]}\to\mathcal{T}^+_{d[k]}\quad\textit{is onto.}$$

In particular, $\mathbb{H}^*_{rel}$ has finite rank over $\mathbb{Z}$. Moreover, one has an exact sequence of finite $\mathbb{Z}$–modules:

$$0\longrightarrow\mathbb{H}^0_{rel}\longrightarrow\mathbb{H}^0_{red}(\Gamma,[k_{a_0}])\oplus\mathbb{Z}^n\longrightarrow\mathbb{H}^0_{red}(\Gamma\setminus j_0,[k])\longrightarrow\mathbb{H}^1_{rel}\longrightarrow\mathbb{H}^1_{red}(\Gamma,[k_{a_0}])\longrightarrow\mathbb{H}^1_{red}(\Gamma\setminus j_0,[k])\cdots$$

where $n:=w_\Gamma(i(k_m)+r_0E^*_{j_0})-\min\{\,w_\Gamma\mid[k_{a_0}]\}\in\mathbb{Z}_{\geq 0}$.

Proof. First note that

$$w_\Gamma(i(k')+r_0E^*_{j_0})=w_{\Gamma\setminus j_0}(k')-\frac{1+r_0^2(E^*_{j_0})^2}{8}.$$

This applied for $k'=k_m$ provides

$$n+\frac{d[k_{a_0}]}{2}=\frac{d[k]}{2}-\frac{1+r_0^2(E^*_{j_0})^2}{8}. \tag{7.2.4}$$

For any $l\geq 0$ and $\bar{k}\in[k_{a_0}]$ set $\bar{\phi}_l\in\mathcal{F}^0(\Gamma,[k_{a_0}])$ defined by

$$\bar{\phi}_l(\bar{k})=U^{-l-d[k_{a_0}]/2+w_L(\bar{k})}\quad(\bar{k}\in[k_{a_0}]).$$

For different $l\geq 0$ they generate $\mathcal{T}^+_{d[k_{a_0}]}\subset\mathbb{H}^0(\Gamma,[k_{a_0}])$. Similarly, set $\{\phi_l\}_{l\geq 0}$ in $\mathcal{F}^0(\Gamma\setminus j_0,[k])$, where

$$\phi_l(k')=U^{-l-d[k]/2+w(k')}\quad(k'\in[k]).$$

They generate $\mathcal{T}^+_{d[k]}$. From the above identities and from the definition of B^0_{rel} one gets for any $l\geq 0$:

$$\mathbb{B}^0_{rel}(\bar{\phi}_{n+l})=\phi_l.$$

Hence the restriction $\mathbb{B}^0_{rel}:\mathcal{T}^+_{d[k_{a_0}]}\to\mathcal{T}^+_{d[k]}$ is onto and the $\mathbb{Z}$–rank of its kernel is n. □

The reader is invited to recall the definition of the Euler characteristic of the lattice cohomology from (2.3.5). We define the *Euler characteristic of the relative lattice cohomology* by

$$eu(\mathbb{H}^*_{rel}):=\sum_{q\geq 0}(-1)^q\,\mathrm{rank}_{\mathbb{Z}}\,\mathbb{H}^q_{rel}.$$

Then the exact sequence of Proposition 7.2.3 and equation (7.2.4) provide

Corollary 7.2.5. *With the notation $r_0:=(i(k_m),E_{j_0})+a_0$ one has*

$$eu(\mathbb{H}^*_{rel}(\Gamma,j_0,[k],a_0))=eu(\mathbb{H}^*(\Gamma,[k_{a_0}]))-eu(\mathbb{H}^*(\Gamma\setminus j_0,[k]))-\frac{1+r_0^2(E^*_{j_0})^2}{8}.$$

Fix any $l\in L(\Gamma\setminus j_0)$, then $k_{a_0}+2l=i(k_m+2l)+r_0E^*_{j_0}$, hence we also get

Corollary 7.2.6. *For any $l\in L(\Gamma\setminus j_0)$ one has*

$$(7.2.7)\qquad \begin{aligned} eu(\mathbb{H}^*_{rel}(\Gamma,j_0,[k_m],a_0)) &= eu(\mathbb{H}^*(\Gamma,[k_{a_0}])) - \frac{(k_{a_0}+2l)^2_\Gamma+|\mathcal{J}|}{8}\\ &-eu(\mathbb{H}^*(\Gamma\setminus j_0,[k_m])) + \frac{(k_m+2l)^2_{\Gamma\setminus j_0}+|\mathcal{J}\setminus j_0|}{8}. \end{aligned}$$

7.3. Reinterpretation. Above, we started with an element k_m of the class $[k]$, and we constructed one of its extensions k_{a_0}. Since $k_{a_0}+i(2l)=i(k_m+2l)+r_0E^*_{j_0}$, we have $k_m+2l=R(k_{a_0}+2l)$ for any $l\in L(\Gamma\setminus j_0)$.

This procedure can be inverted. Indeed, let us fix any $\bar{k}\in Char(\Gamma)$ (which plays the role of $k_{a_0}+i(2l)$). Then set $R(\bar{k})$ and define r_0 by the identity $r_0E^*_{j_0}=\bar{k}-iR(\bar{k})$. Finally, define

$$B_{0,rel}:\mathcal{F}_0(\Gamma\setminus j_0,[R(\bar{k})])\to\mathcal{F}_0(\Gamma,[\bar{k}])\quad\text{by}\quad B_{0,rel}(k')=i(k')+r_0E^*_{j_0},$$

whose kernel is the relative complex $\mathcal{F}^*_{rel}$ with cohomology $\mathbb{H}^*_{rel}(\Gamma,j_0,\bar{k})$. Then (7.2.7) reads as

$$(7.3.1)\qquad \begin{aligned} eu(\mathbb{H}^*_{rel}(\Gamma,j_0,\bar{k})) &= eu(\mathbb{H}^*(\Gamma,[\bar{k}])) - \frac{(\bar{k})^2_\Gamma+|\mathcal{J}|}{8}\\ &-eu(\mathbb{H}^*(\Gamma\setminus j_0,[R(\bar{k})])) + \frac{(R(\bar{k}))^2_{\Gamma\setminus j_0}+|\mathcal{J}\setminus j_0|}{8}. \end{aligned}$$

The identity (7.3.1) depends essentially on the choice of the choice of $\bar{k}\in Char(\Gamma)$. In fact, even if we fix the class $[\bar{k}]$, the choice of the representative $\bar{k}$ from the class $[\bar{k}]$ provides essentially different identities of type (7.3.1): not only the terms $eu(\mathbb{H}^*_{rel}(\Gamma,j_0,\bar{k}))$, $(\bar{k})^2_\Gamma$ and $(R(\bar{k}))^2_{\Gamma\setminus j_0}$ depend on the choice of $\bar{k}$, but even the class $[R(\bar{k})]$.

7.4. The connection with the topological Poincaré series. Let $K_\Gamma\in L'$ denote the *canonical characteristic element* of Γ defined by the *adjunction formulae* $(K_\Gamma+E_j,E_j)+2=0$ for all $j\in\mathcal{J}$. Similarly, one defines $K_{\Gamma\setminus j_0}\in Char(\Gamma\setminus j_0)$. Note that $Char(\Gamma)=K+2L'(\Gamma)$ and $K_{\Gamma\setminus j_0}=R(K_\Gamma)$. The next result computes $eu(\mathbb{H}^*_{rel}(\Gamma,j_0,K+2l'))$ in terms of l' via the coefficients of a series associated with Γ.

Consider the multi-variable Taylor expansion $Z(\mathbf{t})=\sum p_{l'}\mathbf{t}^{l'}$ at the origin of

$$(7.4.1)\qquad \prod_{j\in\mathcal{J}}(1-\mathbf{t}^{E^*_j})^{\delta_j-2},$$

where for any $l'=\sum_j l_jE_j\in L'$ we write $\mathbf{t}^{l'}=\prod_j t_j^{l_j}$, and δ_j is the valency of j. This lives in $\mathbb{Z}[[L']]$, the submodule of formal power series $\mathbb{Z}[[\mathbf{t}^{\pm1/d}]]$ in variables $\{t_j^{\pm1/d}\}_j$, where $d=\det(\Gamma)$. The series $Z(\mathbf{t})$ was used in several articles studying invariants of surface singularities, see [**CDG04, CDG08, CHR04, N08b, N08c, N10**] for different aspects.

For any series $S(\mathbf{t})\in\mathbb{Z}[[L']]$, $S(\mathbf{t})=\sum_{l'}c_{l'}\mathbf{t}^{l'}$, we have the natural decomposition

$$S=\sum_{h\in L'/L}S_h,\quad\text{where}\quad S_h:=\sum_{l'\,:\,[l']=h}c_{l'}\mathbf{t}^{l'}.$$

In particular, for any fixed class $[l']\in L'/L$, one can consider the component $Z_{[l']}(\mathbf{t})$ of $Z(\mathbf{t})$. In fact, see e.g. [**N08b**, (3.1.20)],

$$(7.4.2)\qquad Z_{[l']}(\mathbf{t})=\frac{1}{d}\sum_{\rho\in(L'/L)^\frown}\rho([l'])^{-1}\cdot\prod_{j\in\mathcal{J}}(1-\rho([E^*_j])\mathbf{t}^{E^*_j})^{\delta_j-2},$$

where $(L'/L)^\frown$ is the Pontjagin dual of L'/L.

Furthermore, once the vertex j_0 of Γ is fixed, for any class $[l']\in L'/L$ we set

$$\mathcal{H}_{[l'],j_0}(t):=Z_{[l']}(\mathbf{t})\Big|_{\substack{t_{j_0}=t^d\\ t_j=1\ \text{for}\ j\neq j_0}}\in\mathbb{Z}[[t]].$$

Let $S(t)=\sum_{i\geq0}c_it^i$ be a formal power series. Suppose that for some positive integer p, the expression $\sum_{i=0}^{pn-1}c_i$ is a polynomial $P_p(n)$ in the variable n. Then the constant term of $P_p(n)$ is independent of p. We call this constant term the *periodic constant* of S and denote it by $\mathrm{pc}(S)$ (cf. [**NO09**]).

Proposition 7.4.3. *Fix the vertex j_0 of Γ and write $\mathcal{H}_{[l'],j_0}(t)$ as $\sum_{i\geq0}c_it^i$.*

(a) If $l'=\sum_j a_jE^*_j=\sum_j l'_jE_j\in L'(\Gamma)$ with all a_j sufficiently large then

$$\sum_{i<dl'_{j_0}}c_i=eu(\mathbb{H}^*_{rel}(\Gamma,j_0,K+2l')).$$

(b) Take $\bar{l}' = \sum_j \bar{l}'_j E_j \in L'(\Gamma)$ *with* $\bar{l}'_{j_0} \in [0,1)$*. Then*

$$\mathrm{pc}(\mathcal{H}_{[\bar{l}'],j_0}) = eu(\mathbb{H}^*_{rel}(\Gamma, j_0, K + 2\bar{l}')).$$

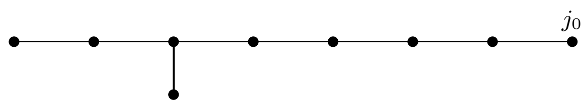
Proof. Use (7.3.1) from above and the identities (3.2.7) and (3.2.13) from [**N10**]. □

7.5. The connection with the Seiberg–Witten invariants. Let $M(\Gamma)$ be the oriented plumbed 3–manifold associated with Γ, and $-M(\Gamma)$ the same 3–manifold with opposite orientation. In is known that the *spin*^{c}–structures of $M(\Gamma)$ (and of $-M(\Gamma)$ too) can be identified with $Char/2L$, see e.g. [**N05, N08**]. Let $\mathbf{sw}(M(\Gamma), [\bar{k}])$ be the Seiberg–Witten invariant of $M(\Gamma)$ associated with the *spin*^{c}–structure $[\bar{k}]$. Then, by Theorem B of the Introduction of [**N10**] for a negative definite graph Γ one has

$$eu(\mathbb{H}^*(\Gamma, [\bar{k}])) = \mathbf{sw}(-M(\Gamma), [\bar{k}]). \tag{7.5.1}$$

Hence the above statements can be reinterpreted in terms of Seiberg–Witten invariants as well.

Example 7.5.2. Let Γ be the next graph, where all the vertices have decoration -2 except the j_0 vertex which has -3.



$\det(\Gamma) = \det(\Gamma \setminus j_0) = 1$, hence for both graphs we have only one class.

Moreover, $\Gamma \setminus j_0$ is rational (an E_8–graph), hence $\min w_{\Gamma\setminus j_0} = 0$, $\mathbb{H}^*_{red}(\Gamma \setminus j_0) = 0$ and $eu(\mathbb{H}^*(\Gamma \setminus j_0)) = 0$.

On the other hand, Γ is minimally elliptic, $\min w_\Gamma = 0$, $\mathbb{H}^*_{red}(\Gamma) = \mathbb{H}^0_{red}(\Gamma) = \mathbb{Z}_{(0)}$, the rank one $\mathbb{Z}$–module concentrated at degree zero. Hence $eu(\mathbb{H}^*(\Gamma)) = 1$.

By the long exact sequence, we get $\mathbb{H}^q_{rel}(\Gamma, j_0, \bar{k}) = 0$ for any $\bar{k}$ and $q > 0$.

It is easy to see that $K_\Gamma = -E^*_{j_0}$, and $(E^*_{j_0})^2 = -1$. Therefore, if $\bar{k} = K + 2l'$, and $r_0 := -(\bar{k}, E^*_{j_0})$ and $l'_{j_0} := -(l', E^*_{j_0})$, then $r_0 = 2l'_{j_0} - 1$. By (7.2.5) or (7.3.1)

$$\mathrm{rank}\, \mathbb{H}^0_{rel}(\Gamma, j_0, \bar{k}) = eu(\mathbb{H}^*(\Gamma, j_0, \bar{k})) = \frac{r_0^2 + 7}{8} = 1 + \frac{l'_{j_0}(l'_{j_0} - 1)}{2}.$$

By a computation one obtains

$$\mathcal{H}_{[0],j_0}(t) = \frac{1 - t^6}{(1-t^3)(1-t^2)(1-t)} = \frac{1 - t + t^2}{(1-t)^2} = 1 + t + 2t^2 + 3t^3 + 4t^4 + \cdots.$$

Then

$$\sum_{i < l'_{j_0}} c_i = 1 + 1 + 2 + 3 + \cdots + (l'_{j_0} - 1) = 1 + \frac{l'_{j_0}(l'_{j_0} - 1)}{2},$$

hence (7.4.3)(a) follows. In order to exemplify part (b), we have to take $\bar{l}'$ with $\bar{l}'_{j_0} = 0$, hence in this case $eu(\mathbb{H}^*(\Gamma, j_0, \bar{k})) = 1$. But one also has

$$\mathrm{pc}\ \frac{1 - t + t^2}{(1-t)^2} = 1.$$

Example 7.5.3. Assume that Γ is a star–shaped graph with central vertex j_0 and we fix $\bar{k} = K_\Gamma$.

Since all the connected components of $\Gamma \setminus j_0$ are strings (i.e. rational graphs), for them (cf. [**N08**])

$$eu(\mathbb{H}^*(\Gamma \setminus j_0, [K_{\Gamma\setminus j_0}])) - \frac{(K_{\Gamma\setminus j_0})^2 + |\mathcal{J} \setminus j_0|}{8} = 0.$$

Moreover, $\mathbb{H}^q(\Gamma, [K]) = \mathbb{H}^q_{rel}(\Gamma, j_0, [K]) = 0$ for $q > 0$, and

$$\begin{aligned}\mathrm{rank}\, \mathbb{H}^0_{rel}(\Gamma, j_0, [K]) = eu(\mathbb{H}^*(\Gamma, [K])) - \frac{K^2 + |\mathcal{J}|}{8} &= \mathrm{rank}\, \mathbb{H}^0(\Gamma, [K]) - \min w_\Gamma - \frac{K^2 + |\mathcal{J}|}{8} \\ &= \mathrm{rank}\, \mathbb{H}^0(\Gamma, [K]) - \min \chi_K.\end{aligned} \tag{7.5.4}$$

This equals the periodic constant of $\mathcal{H}_{[0],j_0}(t)$ by [**N08, BN10**]. Moreover, if (X, o) is a weighted homogeneous normal surface singularity with minimal good resolution graph Γ, then its *geometric genus* p_g equals the last term of (7.5.4), cf. e.g. [**NN04, N05**]. Hence $p_g(X, o) = \mathrm{rank}\, \mathbb{H}^0_{rel}(\Gamma, j_0, [K])$.

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